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INFLUENCE OF A NON-NODAL POINT IN THE ADJUSTMENT OF LOCAL LEVELLING NETWORKS

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ABSTRACT

A local levelling network is composed of bench marks (points), which can be connected into topology loops. These points can be nodal or non-nodal points and have different influence in the adjustment of a local levelling network. Non-nodal points can be excluded from the adjustment by which we achieve better overview of relevant information about a levelling network. By excluding non-nodal points from the adjustment the number of normal equations is reduced. Non-nodal points can be included in the adjustment if we want to get information about non-nodal points. Calculated values, accuracy and functions of nodal points are identical to those we get by including nodal points in the adjustment of a local levelling network. When estimating the quality for field monitoring, which is under the influence of mining, it is better to include only nodal (relevant) points.

Influence of a non-nodal point in the adjustment of local levelling networks is presented in the article.

Key words: Bench mark (point), nodal point, non-nodal point, levelling network, network adjustment.

INTRODUCTION

Local levelling networks (of smaller size) are adjusted by parameter variaton model of the method of the least squares. In the network there are points (bench marks) that may be classified into nodal and non-nodal points, in topological sense. The exclusion of non-nodal points that are locally not over determined can be excluded from adjusment. In that case the sistem of normal equations is evidently smaller. In the paper there is given a proof that the ellimination of non-nodal points does not influense the adjusment results. It will be illustrated by numerical examples, too.

DEFINITION OF A NODAL AND A NON-NODAL POINT

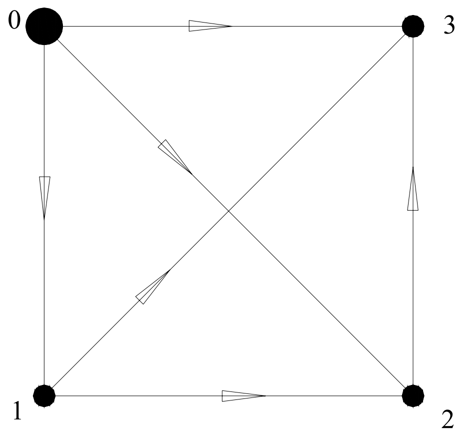

Figure 1.: Local levelling network.

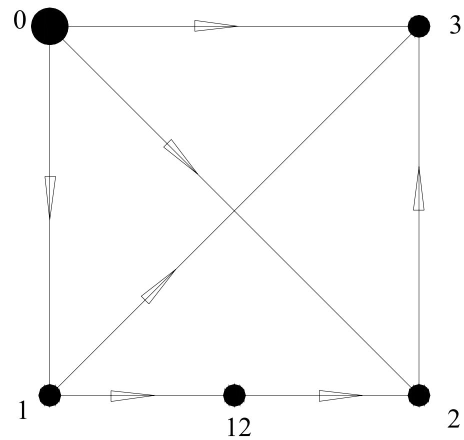

Figure 2.: Local levelling network.

Figures 1 and 2 show two local levelling networks of bench marks (points) with measured height differences. Figure 1 consists of nodal points **0**, **1**, **2** and **3**. Figure 2 consists of nodal points **0**, **1**, **2** , **3** and a non-nodal point **12**.

A nodal point has three or more connections to other points in a local network. A non-nodal point has less than three connections to other points in a local levelling network. In this case, it is point **12** that has only two connections (to points **1** and **2**).

ADJUSTMENT OF A LOCAL LEVELLING NETWORK THEORY

Figure 3 shows a local levelling network with points **0**, **1**, **2, 3** and **12**. Point **0** is the datum point, nodal points **1**, **2, 3** and non-nodal point **12** are unknown points.

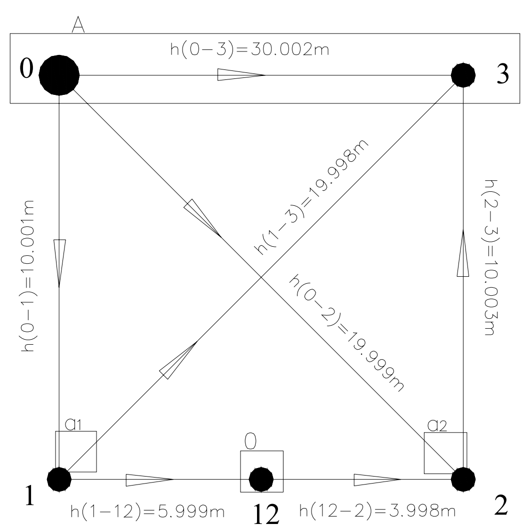

Figure 3.: Local levelling network with measured altitude differences.

The standard form of the observation equations of the Gauss-Markov linear model model (1) in a standard matrix form

$$\bar{\mathbf{v}} = \bar{\mathbf{A}} \cdot \bar{\mathbf{x}} + \bar{\mathbf{f}} \leftarrow \bar{\mathbf{Q}}_{ll} \qquad (1)$$

Are written as

$$\bar{\mathbf{v}} = \begin{bmatrix} \bar{\mathbf{A}} & \bar{\mathbf{f}} \end{bmatrix} \cdot \begin{bmatrix} \bar{\mathbf{x}} \\ 1 \end{bmatrix} \leftarrow \begin{bmatrix} \bar{\mathbf{Q}}_{ll} \end{bmatrix} \qquad (2)$$

where

$\bar{\mathbf{v}}$ - vector of residuals (n rows)

$\bar{\mathbf{A}}$ - incidence (design) matrix (n rows, u columns)

$\bar{\mathbf{f}}$ - vector of absolute terms (n rows)

The matrix $\bar{\mathbf{A}}$ is decomposed into two columns $\mathbf{a}_1$ and $\mathbf{a}_2$, and the residual matrix $\mathbf{A}$ (Fig. 3)

The network design matrix $\bar{\mathbf{A}}$ is partitioned according to its future use if the non-nodal point **12** is eliminated:

1. The point **12** is linked to points **1** and **2** by the two last observation equations that are cut off the $\bar{\mathbf{A}}$
2. The two columns of **1** and **2** of the letter $\bar{\mathbf{A}}$ are considered as independant $\mathbf{a}_1$ and $\mathbf{a}_2$ giving the connections of the points **1**-**2** to the remaining network points (but **1**, **2** and **12**).
3. The column of **12** is evidently a zero vector (as **12** is connected to no other point of the network) as **12** is a non-nodal point.
4. What is half of $\bar{\mathbf{A}}$ is now stored in $\mathbf{A}$.
5. Accordingly, the vector of unknowns $\mathbf{X}$ contains the unknowns of $\mathbf{A}$ and the unknowns for **1** and **2** that are designated as Y_1, Y_2 and Z for **12**.
6. The $\bar{\mathbf{Q}}_{ll}$ is partitioned into $\mathbf{Q}_{ll}$ for observations in $\mathbf{A}$, m_1^2 and m_2^2 for the two last observation equations of $\bar{\mathbf{A}}$.

$$\begin{bmatrix} \underset{n\times 1}{\mathbf{v}} \\ v_1 \\ v_2 \end{bmatrix} = \left[\begin{array}{ccccc|c} \underset{n\times u}{\mathbf{A}} & \underset{n\times 1}{\mathbf{a}_1} & \underset{n\times 1}{\mathbf{a}_2} & \underset{n\times 1}{\mathbf{0}} & & \underset{n\times 1}{\mathbf{f}} \\ \underset{1\times u}{\mathbf{0}} & -1 & 0 & 1 & & f_1 \\ \underset{1\times u}{\mathbf{0}} & 0 & 1 & -1 & & f_2 \end{array}\right] \cdot \begin{bmatrix} \underset{u\times 1}{\mathbf{X}} \\ Y_1 \\ Y_2 \\ Z \\ \hline 1 \end{bmatrix} \leftarrow \begin{bmatrix} \underset{n\times n}{\mathbf{Q}_{ll}} & \underset{n\times 1}{\mathbf{0}} & \underset{n\times 1}{\mathbf{0}} \\ \underset{1\times n}{\mathbf{0}} & m_1^2 & 0 \\ \underset{n\times 1}{\mathbf{0}} & 0 & m_2^2 \end{bmatrix} \qquad (3)$$

$$\begin{bmatrix} \underset{u+2\times 1}{\xi} \end{bmatrix} = \begin{bmatrix} \underset{u\times 1}{\mathbf{X}} \\ Y_1 \\ Y_2 \end{bmatrix} \qquad (4)$$

The condition $w_z = \underset{1\times n}{\mathbf{v}^T} \underset{n\times n}{\mathbf{Q}_{ll}^{-1}} \underset{n\times 1}{\mathbf{v}} + \frac{v_1^2}{m_1^2} + \frac{v_2^2}{m_2^2} = \min$ gives

$$\left[\begin{array}{ccccc|c} \underset{u\times n}{\mathbf{A}^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times u}{\mathbf{A}} & \underset{u\times n}{\mathbf{A}^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times 1}{\mathbf{a}_1} & \underset{u\times n}{\mathbf{A}^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times 1}{\mathbf{a}_2} & \underset{u\times 1}{\mathbf{0}} & & \underset{u\times n}{\mathbf{A}^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times 1}{\mathbf{f}} \\ \underset{1\times n}{\mathbf{a}_1^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times u}{\mathbf{A}} & \underset{1\times n}{\mathbf{a}_1^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times 1}{\mathbf{a}_1}+\frac{1}{m_1^2} & \underset{1\times n}{\mathbf{a}_1^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times 1}{\mathbf{a}_2} & \frac{-1}{m_1^2} & & \underset{1\times n}{\mathbf{a}_1^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times 1}{\mathbf{f}}-\frac{f_1}{m_1^2} \\ \underset{1\times n}{\mathbf{a}_2^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times u}{\mathbf{A}} & \underset{1\times n}{\mathbf{a}_2^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times 1}{\mathbf{a}_1} & \underset{1\times n}{\mathbf{a}_2^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times 1}{\mathbf{a}_2}+\frac{1}{m_2^2} & \frac{-1}{m_2^2} & & \underset{1\times n}{\mathbf{a}_2^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times 1}{\mathbf{f}}+\frac{f_2}{m_2^2} \\ \underset{1\times u}{\mathbf{0}} & \frac{-1}{m_1^2} & \frac{-1}{m_2^2} & \frac{1}{m_1^2}+\frac{1}{m_2^2} & & \frac{f_1}{m_1^2}-\frac{f_2}{m_2^2} \\ \hline \underset{1\times n}{\mathbf{f}^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times u}{\mathbf{A}} & \underset{1\times n}{\mathbf{f}^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times 1}{\mathbf{a}_1}-\frac{f_1}{m_1^2} & \underset{1\times n}{\mathbf{f}^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times 1}{\mathbf{a}_2}+\frac{f_2}{m_2^2} & \frac{f_1}{m_1^2}-\frac{f_2}{m_2^2} & & \underset{1\times n}{\mathbf{f}^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times 1}{\mathbf{f}}+\frac{f_1}{m_1^2}+\frac{f_2}{m_2^2} \end{array}\right] = \left[\begin{array}{c|c} \mathbf{N_{xx}} & \mathbf{n} \\ \hline \mathbf{n}^T & \mathbf{f}^T\mathbf{Pf} \end{array}\right] \qquad (5)$$

In theory $\mathbf{N}_{xx}\xi = \mathbf{n}$

By inverting the matrix (5) and left multiplication, we get the results:

$$\underset{u+2\times 1}{\xi} = -\underset{u+2\times u+2}{\mathbf{Q}}\ \underset{u+2\times 1}{\mathbf{n}} \qquad (6)$$

$$Z = \frac{\frac{Y_1 - f_1}{m_1^2} - \frac{Y_2 + f_2}{m_2^2}}{\frac{1}{m_1^2} + \frac{1}{m_2^2}} = \frac{(Y_1 - f_1)p_1 + (Y_2 + f_2)p_2}{p_1 + p_2} \qquad (7)$$

Z is value for a non-nodal point **12**.

$$\underset{u+2\times u+2}{\mathbf{Q}_{\xi\xi}} = \underset{u+2\times u+2}{\mathbf{Q}} \qquad (8)$$

$$\underset{u+2\times u+2}{\mathbf{Q}_{\xi Z}} = \underset{u+2\times u+2}{\mathbf{Q}} \frac{\begin{bmatrix} \underset{u\times 1}{\mathbf{0}} \\ m_2^2 \\ m_1^2 \end{bmatrix}}{m^2} = \underset{u+2\times u+2}{\mathbf{Q}_{\xi Z}^T} \qquad (9)$$

$$\underset{u+2\times u+2}{\mathbf{Q}_{ZZ}} = \frac{m_1^2 m_2^2}{m^2} + \frac{\begin{bmatrix} m_2^2 & m_1^2 \end{bmatrix}}{m^2} \begin{bmatrix} Q_{Y_1Y_1} & Q_{Y_1Y_2} \\ Q_{Y_1Y_2} & Q_{Y_2Y_2} \end{bmatrix} \frac{\begin{bmatrix} m_2^2 \\ m_1^2 \end{bmatrix}}{m^2} \qquad (10)$$

$$\underset{u+2\times u+2}{\mathbf{Q}_{ZZ}} = \frac{1}{\frac{1}{m_2^2} + \frac{1}{m_1^2}} + \frac{\frac{Q_{Y_1Y_1}}{m_1^2 m_1^2} + 2\frac{Q_{Y_1Y_2}}{m_1^2 m_2^2} + \frac{Q_{Y_2Y_2}}{m_1^2 m_1^2}}{\left(\frac{1}{m_2^2} + \frac{1}{m_1^2}\right)^2} = \frac{1}{p_2 + p_1} + \frac{Q_{Y_1Y_1} p_1^2 + 2Q_{Y_1Y_1} p_1 p_2 + Q_{Y_2Y_2} p_2^2}{(p_2 + p_1)^2} \qquad (11)$$

$$w_Z = F - \underset{1\times u+2}{\mathbf{n}^T}\ \underset{u+2\times u+2}{\mathbf{Q}}\ \underset{u+2\times 1}{\mathbf{n}} = w \qquad (12)$$

$$w_Z = w = \underset{1\times n}{\mathbf{v}^T}\ \underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\ \underset{n\times 1}{\mathbf{v}} \qquad (13)$$

EXAMPLE (CALCULATION)

The following data are shown in the columns of table 1.:

- Name of the point,
- Approximate heights of points (m).

Table 1.:

POINT	H(m)
0	0
3	30
1	10
2	20
12	16

Table 2. shows the matrix $\mathbf{A}$. The matrix $\mathbf{A}$ gives the geometry of a local levelling network. It is composed of elements with values -1, 1 and 0. -1 represents the beginning of the measurement, 1 represents the end of the measurement and 0 means that the point is not included in the measurement.

Table 2.:

			A	a1	a2	0
	A	**0**	**3**	**1**	**2**	**12**
0	**1**	-1	0	1	0	0
0	**2**	-1	0	0	1	0
0	**3**	-1	1	0	0	0
1	**3**	0	1	-1	0	0
2	**3**	0	1	0	-1	0
1	**12**	0	0	-1	0	1
12	**2**	0	0	0	1	-1

Here are the values of individual terms, which are included in the calculation. The same values are shown in tables 2 and 3.

$$\underset{n\times u}{\mathbf{A}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \underset{n\times 1}{\mathbf{a}_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \qquad \underset{n\times 1}{\mathbf{a}_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \qquad \underset{n\times 1}{\mathbf{f}} = \begin{bmatrix} -0.001 \\ 0.001 \\ -0.002 \\ 0.002 \\ 0.003 \end{bmatrix}$$

$$\underset{n\times n}{\mathbf{Q}_{ll}} = \begin{bmatrix} 10 & & & & \\ & 20 & & & \\ & & 10 & & \\ & & & 20 & \\ & & & & 10 \end{bmatrix} \qquad \underset{1\times u}{\mathbf{0}} = 0 \qquad \begin{aligned} f_1 &= 0.001 \\ f_2 &= 0.002 \end{aligned}$$

$$\begin{aligned} \underset{u\times 1}{\mathbf{X}} &= [x_3] \\ Y_1 &= x_1 \\ Y_2 &= x_2 \\ Z &= x_{12} \end{aligned} \qquad \begin{aligned} m_1 &= 6 \\ m_2 &= 8 \end{aligned} \qquad \underset{n\times 1}{\mathbf{v}} = \begin{bmatrix} v_{0-1} \\ v_{0-2} \\ v_{0-3} \\ v_{1-3} \\ v_{2-3} \end{bmatrix}$$

The following data are shown in the collumns of table 3:

- Δh_{apx} – approximate altitude differences between points (m)
- L – measured altitude differences between points (m)

- f – absolute term vector (mm)
- m – mean square root error (mm)
- p – veights (mm^{-2})

Table 3.:

		Δh **apx**	**L**	**f**	**m^2**	**m**	**p**
0	**1**	10	10.001	-0.001	100	10	0.01
0	**2**	20	19.999	0.001	400	20	0.0025
0	**3**	30	30.002	-0.002	100	10	0.01
1	**3**	20	19.998	0.002	400	20	0.0025
2	**3**	10	10.003	-0.003	100	10	0.01
1	**2**	10	9.997	0.003	100	10	0.01
1	**12**	6	5.999	0.001	36	6	0.027777778
12	**2**	4	3.998	0.002	64	8	0.015625

Table 4. shovs the matrix $\mathbf{A}_{1\to12\to2}$.

Table 4.:

	$A_{1\to12\to2}$	**0**	**3**	**1**	**2**	**12**	**f**
0	**1**		0	0.1	0	0	-0.0001
0	**2**		0	0	0.05	0	0.0001
0	**3**		0.1	0	0	0	-0.0002
1	**3**		0.05	-0.05	0	0	0.0001
2	**3**		0.1	0	-0.1	0	-0.0003
1	**12**		0	-0.1666666	0	0.166666667	0.0002
12	**2**		0	0	0.125	-0.125	0.0002

Table 5. shows the matrix $\mathbf{N}_{1\to12\to2}$ and vector **n**.

Table 5.:

$N_{1\to12\to2}$	**0**	**3**	**1**	**2**	**12**	**n**
0						
3		0.0225	-0.0025	-0.01	0	-4.5E-05
1		-0.0025	0.040277778	0	-0.02777777	-4.27778E-05
2		-0.01	0	0.028125	-0.015625	6.375E-05
12		0	-0.027777778	-0.015625	0.043402778	-3.47222E-06
n		-0.000045	-0.000043	0.000064	-0.000003	2.42778E-07

$$\mathbf{N}_{1\to12\to2} = \mathbf{A}^T \mathbf{Q}_{ll}^{-1} \mathbf{A} = \mathbf{A}^T \mathbf{P} \mathbf{A} \tag{14}$$

$\mathbf{N}_{1\to12\to2}$ – matrix of normal equations

$$\mathbf{n} = \mathbf{A}^T \mathbf{P} \mathbf{f} \tag{15}$$

$\mathbf{n}$ – absolute term of normal equations

Table 6. shows the matrix $\mathbf{Q}_{1\rightarrow 12\rightarrow 2}$ and the vector $\mathbf{X}$.

Table 6.:

Q1→12→2	**0**	**3**	**1**	**2**	**12**	**x**
0						
3		65	25	40	30.4	0.00155
1		25	65	40	56	0.00155
2		40	40	80	54.4	-0.0014
12		30.4	56	54.4	78.464	0.000568

$$\mathbf{Q}_{1\rightarrow 12\rightarrow 2} = \mathbf{N}^{-1}_{1\rightarrow 12\rightarrow 2} \tag{16}$$

$$\mathbf{X} = -\mathbf{Q}_{1\rightarrow 12\rightarrow 2}\mathbf{n} \tag{17}$$

X – vector of unknowns (solution)

ADJUSTMENT OF A LOCAL LEVELLING NETWORK OF NODAL POINTS ONLY THEORY

Figure 4. shows a local levelling network with points **0**, **1**, **2** and **3**. Point **0** is the datum point, nodal points **1**, **2** and **3** are unknown (variable) points.

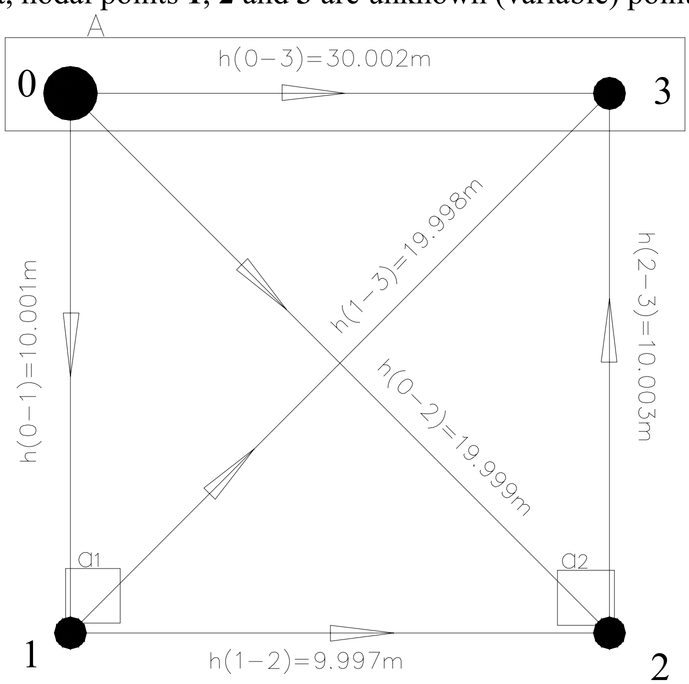

Figure 4.: Local levelling network with measured altitude differences

$$\begin{bmatrix} \underset{n\times 1}{\mathbf{v}} \\ v = v_1 + v_2 \end{bmatrix} = \left[\begin{array}{ccc|c} \underset{n\times u}{\mathbf{A}} & \underset{n\times 1}{\mathbf{a}_1} & \underset{n\times 1}{\mathbf{a}_2} & \underset{n\times 1}{\mathbf{f}} \\ \underset{1\times u}{\mathbf{0}} & -1 & 1 & f = f_1 + f_2 \end{array}\right] \begin{bmatrix} \underset{u\times 1}{\mathbf{X}} \\ Y_1 \\ Y_2 \\ \hline 1 \end{bmatrix} \leftarrow \begin{bmatrix} \underset{n\times n}{\mathbf{Q}_{ll}} & \underset{n\times 1}{\mathbf{0}} \\ \underset{1\times n}{\mathbf{0}} & m^2 = m_1^2 + m_2^2 \end{bmatrix} \tag{18}$$

The condition $w = \underset{1\times n}{\mathbf{v}^T} \underset{n\times n}{\mathbf{Q}_{ll}^{-1}} \underset{n\times 1}{\mathbf{v}} + \dfrac{v^2}{m^2} = \min$ gives

$$\left[\begin{array}{ccc|c}
\underset{u\times n}{\mathbf{A}^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times u}{\mathbf{A}} & \underset{u\times n}{\mathbf{A}^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times 1}{\mathbf{a}_1} & \underset{u\times n}{\mathbf{A}^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times 1}{\mathbf{a}_2} & \underset{u\times n}{\mathbf{A}^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times 1}{\mathbf{f}} \\
\underset{1\times n}{\mathbf{a}_1^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times u}{\mathbf{A}} & \underset{1\times n}{\mathbf{a}_1^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times 1}{\mathbf{a}_1}+\dfrac{1}{m^2} & \underset{1\times n}{\mathbf{a}_1^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times 1}{\mathbf{a}_2}-\dfrac{1}{m^2} & \underset{1\times n}{\mathbf{a}_1^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times 1}{\mathbf{f}}-\dfrac{f}{m^2} \\
\underset{1\times n}{\mathbf{a}_2^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times u}{\mathbf{A}} & \underset{1\times n}{\mathbf{a}_2^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times 1}{\mathbf{a}_1}-\dfrac{1}{m^2} & \underset{1\times n}{\mathbf{a}_2^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times 1}{\mathbf{a}_2}+\dfrac{1}{m_2^2} & \underset{1\times n}{\mathbf{a}_2^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times 1}{\mathbf{f}}+\dfrac{f}{m^2} \\
\hline
\underset{1\times n}{\mathbf{f}^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times u}{\mathbf{A}} & \underset{1\times n}{\mathbf{f}^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times 1}{\mathbf{a}_1}-\dfrac{f}{m^2} & \underset{1\times n}{\mathbf{f}^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times 1}{\mathbf{a}_2}+\dfrac{f}{m^2} & \underset{1\times n}{\mathbf{f}^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times 1}{\mathbf{f}}+\dfrac{f^2}{m^2}
\end{array}\right]
\left[\begin{array}{c}\underset{u\times 1}{\mathbf{X}} \\ Y_1 \\ Y_2 \\ \hline 1\end{array}\right] = 0 \tag{19}$$

$$\left[\begin{array}{ccc|c}
\underset{u\times u}{\mathbf{N}_{\mathbf{XX}}} & \underset{u\times 1}{\mathbf{N}_{\mathbf{X}Y_1}} & \underset{u\times 1}{\mathbf{N}_{\mathbf{X}Y_2}} & \underset{u\times 1}{\mathbf{n}_{\mathbf{X}}} \\
\underset{1\times u}{\mathbf{N}_{\mathbf{X}Y_1}^T} & N_{Y_1Y_1} & N_{Y_1Y_2} & n_{Y_1} \\
\underset{1\times u}{\mathbf{N}_{\mathbf{X}Y_2}^T} & N_{Y_1Y_2} & N_{Y_2Y_2} & n_{Y_2} \\
\hline
\underset{1\times u}{\mathbf{n}_{\mathbf{X}}^T} & n_{Y_1} & n_{Y_2} & F
\end{array}\right]
=
\left[\begin{array}{ccc|c}
\underset{u\times n}{\mathbf{A}^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times u}{\mathbf{A}} & \underset{u\times n}{\mathbf{A}^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times 1}{\mathbf{a}_1} & \underset{u\times n}{\mathbf{A}^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times 1}{\mathbf{a}_2} & \underset{u\times n}{\mathbf{A}^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times 1}{\mathbf{f}} \\
\underset{1\times n}{\mathbf{a}_1^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times u}{\mathbf{A}} & \underset{1\times n}{\mathbf{a}_1^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times 1}{\mathbf{a}_1}+\dfrac{1}{m^2} & \underset{1\times n}{\mathbf{a}_1^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times 1}{\mathbf{a}_2}-\dfrac{1}{m^2} & \underset{1\times n}{\mathbf{a}_1^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times 1}{\mathbf{f}}-\dfrac{f}{m^2} \\
\underset{1\times n}{\mathbf{a}_2^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times u}{\mathbf{A}} & \underset{1\times n}{\mathbf{a}_2^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times 1}{\mathbf{a}_1}-\dfrac{1}{m^2} & \underset{1\times n}{\mathbf{a}_2^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times 1}{\mathbf{a}_2}+\dfrac{1}{m_2^2} & \underset{1\times n}{\mathbf{a}_2^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times 1}{\mathbf{f}}+\dfrac{f}{m^2} \\
\hline
\underset{1\times n}{\mathbf{f}^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times u}{\mathbf{A}} & \underset{1\times n}{\mathbf{f}^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times 1}{\mathbf{a}_1}-\dfrac{f}{m^2} & \underset{1\times n}{\mathbf{f}^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times 1}{\mathbf{a}_2}+\dfrac{f}{m^2} & \underset{1\times n}{\mathbf{f}^T}\underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\underset{n\times 1}{\mathbf{f}}+\dfrac{f^2}{m^2}
\end{array}\right] \tag{20}$$

$$\left[\begin{array}{ccc|c}
\underset{u\times u}{\mathbf{N}_{\mathbf{XX}}} & \underset{u\times 1}{\mathbf{N}_{\mathbf{X}Y_1}} & \underset{u\times 1}{\mathbf{N}_{\mathbf{X}Y_2}} & \underset{u\times 1}{\mathbf{n}_{\mathbf{X}}} \\
\underset{1\times u}{\mathbf{N}_{\mathbf{X}Y_1}^T} & N_{Y_1Y_1} & N_{Y_1Y_2} & n_{Y_1} \\
\underset{1\times u}{\mathbf{N}_{\mathbf{X}Y_2}^T} & N_{Y_1Y_2} & N_{Y_2Y_2} & n_{Y_2} \\
\hline
\underset{1\times u}{\mathbf{n}_{\mathbf{X}}^T} & n_{Y_1} & n_{Y_2} & F
\end{array}\right]
\left[\begin{array}{c}\underset{u\times 1}{\mathbf{X}} \\ Y_1 \\ Y_2 \\ \hline 1\end{array}\right] = 0 \tag{21}$$

The matrix should be partitioned to more parts for better clearness and easier understanding. That enables easier calculations.

$$\underset{u+2\times u+2}{\mathbf{N}} = \begin{bmatrix}
\underset{u\times u}{\mathbf{N}_{\mathbf{XX}}} & \underset{u\times 1}{\mathbf{N}_{\mathbf{X}Y_1}} & \underset{u\times 1}{\mathbf{N}_{\mathbf{X}Y_2}} \\
\underset{1\times u}{\mathbf{N}_{\mathbf{X}Y_1}^T} & N_{Y_1Y_1} & N_{Y_1Y_2} \\
\underset{1\times u}{\mathbf{N}_{\mathbf{X}Y_2}^T} & N_{Y_1Y_2} & N_{Y_2Y_2}
\end{bmatrix} \tag{22}$$

$$\underset{u+2\times 1}{\mathbf{n}} = \begin{bmatrix} \underset{u\times 1}{\mathbf{n}_{\mathbf{X}}} \\ n_{Y_1} \\ n_{Y_2} \end{bmatrix} \qquad \underset{1\times u}{\mathbf{n}^T} = \begin{bmatrix} \underset{1\times u}{\mathbf{n}_{\mathbf{X}}^T} & n_{Y_1} & n_{Y_2} \end{bmatrix} \qquad F \qquad \underset{u+2\times 1}{\xi} = \begin{bmatrix} \underset{u\times 1}{\mathbf{X}} \\ Y_1 \\ Y_2 \end{bmatrix}$$

$$\left[\begin{array}{c|c} \underset{u+2\times u+2}{\mathbf{N}} & \underset{u+2\times 1}{\mathbf{n}} \\ \hline \underset{1\times u}{\mathbf{n}^T} & F \end{array}\right] \left[\begin{array}{c} \underset{u+2\times 1}{\xi} \\ \hline 1 \end{array}\right] = 0 \tag{23}$$

After inverting the matrix, we get the values for $\underset{u+2\times 1}{\xi}$ and w.

$$\underset{u+2\times 1}{\xi} = -\underset{u+2\times u+2}{\mathbf{Q}}\,\underset{u+2\times 1}{\mathbf{n}} \tag{24}$$

$$w = F - \underset{1\times u+2}{\mathbf{n}^T}\ \underset{u+2\times u+2}{\mathbf{Q}}\ \underset{u+2\times 1}{\mathbf{n}} \tag{25}$$

The proof that a non-nodal point has no influence in the adjustment of a local levelling network is as follows: $w_z = w$

$$w = \underset{1\times n}{\mathbf{v}^T}\ \underset{n\times n}{\mathbf{Q}_{ll}^{-1}}\ \underset{n\times 1}{\mathbf{v}} \tag{26}$$

$$\underset{u+2\times u+2}{\mathbf{Q}_{\xi\xi}} = \underset{u+2\times u+2}{\mathbf{Q}} = \begin{bmatrix} \underset{u\times u}{\mathbf{Q}_{\mathbf{XX}}} & \underset{u\times 1}{\mathbf{Q}_{\mathbf{X}Y_1}} & \underset{u\times 1}{\mathbf{Q}_{\mathbf{X}Y_2}} \\ \underset{1\times u}{\mathbf{Q}_{\mathbf{X}Y_1}^T} & Q_{Y_1Y_1} & Q_{Y_1Y_2} \\ \underset{1\times u}{\mathbf{Q}_{\mathbf{X}Y_2}^T} & Q_{Y_1Y_2} & Q_{Y_2Y_2} \end{bmatrix} \tag{27}$$

EXAMPLE (CALCULATION)

The following data are shown in the columns of table 7.:

- Name of the point,
- Approximate heights of points (m).

Table 7.:

POINT	**H(m)**
0	0
3	30
1	10
2	20

Table 8. shows the matrix **A**. The matrix **A** gives the geometry of a local levelling network. It is composed of elements with values -1, 1 and 0. -1 represents the beginning of the measurement, 1 represents the end of the measurement and 0 means that the point is not included in the measurement.

Table 8.:

			A	a_1	a_2
	A	**0**	**3**	**1**	**2**
0	**1**	-1	0	1	0
0	**2**	-1	0	0	1
0	**3**	-1	1	0	0
1	**3**	0	1	-1	0
2	**3**	0	1	0	-1
1	**2**	0	0	-1	1

The following data are shown in the collumns of table 9.:

- Δh_{apx} – approximate altitude differences between points (m)
- L – measured altitude differences between points (m)
- f – absolute term vector (mm)

- m – mean square root error (mm)
- p – veights (mm^{-2})

Table 9.:

		Δh **apx**	**L**	**f**	$\mathbf{m^2}$	**m**	**p**
0	**1**	10	10.001	-0.001	100	10	0.01
0	**2**	20	19.999	0.001	400	20	0.0025
0	**3**	30	30.002	-0.002	100	10	0.01
1	**3**	20	19.998	0.002	400	20	0.0025
2	**3**	10	10.003	-0.003	100	10	0.01
1	**2**	10	9.997	0.003	100	10	0.01

Table 4. shovs the matrix $\mathbf{A_{1\rightarrow 2}}$.

Table 10.:

	$\mathbf{A_{1\rightarrow 2}}$	**0**	**3**	**1**	**2**	**f**
0	**1**		0	0.1	0	-0.0001
0	**2**		0	0	0.05	0.0001
0	**3**		0.1	0	0	-0.0002
1	**3**		0.05	-0.05	0	0.0001
2	**3**		0.1	0	-0.1	-0.0003
1	**2**		0	-0.1	0.1	0.0003

Table 10 shows the matrix $\mathbf{N_{1\rightarrow 2}}$ and vector **n**.

Table 11.:

$\mathbf{N_{1\rightarrow 2}}$	**0**	**3**	**1**	**2**	**n**
0					
3		0.0225	-0.0025	-0.01	-0.000045
1		-0.0025	0.0225	-0.01	-0.000045
2		-0.01	-0.01	0.0225	0.0000625
n		-0.000045	-0.000045	0.0000625	0.0000002425

$$\mathbf{N}_{1\rightarrow 2} = \mathbf{A}^T \mathbf{Q}_{ll}^{-1} \mathbf{A} = \mathbf{A}^T \mathbf{P} \mathbf{A} \tag{28}$$

$\mathbf{N}_{1\rightarrow 2}$ – matrix of normal equations

$$\mathbf{n} = \mathbf{A}^T \mathbf{P} \mathbf{f} \tag{29}$$

$\mathbf{n}$ – absolute term of normal equations

Table 12. shows the matrix $\mathbf{Q}_{1\rightarrow 2}$ and the vector $\mathbf{X}$.

Table 12.:

Q1→2	**0**	**3**	**1**	**2**	**x**
0					
3		65	25	40	0.00155
1		25	65	40	0.00155
2		40	40	80	-0.0014

COMPARISON OF RESULTS

Table 13. shows values of **X** and adjusted heights of points **0**, **1**, **2, 3** and **12** H_{adj} with and without a non-nodal point **12**.

Table 13.:

		x	x	H_{adj}	H_{adj}
		without	with	without	with
POINT	**H(m)**	**X**, Y_1, Y_2	**X**, Y_1, Y_2	**X**, Y_1, Y_2	**X**, Y_1, Y_2
0	0				
3	30	0.00155	-0.00155	30.00155	0.00155
1	10	0.00155	-0.00155	10.00155	0.00155
2	20	-0.0014	0.0014	19.9986	-0.0014
12	16		-0.000568		0.000568

Table 14. shows calculated values of Z without a non-nodal point **12**. Table 15 shows calculated values of Z with a non-nodal point **12**. Both values are identical.

Table 14.:

$(y_1-f_1)*p_1=$	0.00001528
$(y_2+f_2)*p_2=$	0.00000937
$(y_1-f_1)*p_1+(y_2+f_2)*p_2=$	0.00002465
$(y_1-f_1)*p_1+(y_2+f_2)*p_2/(p_1+p_2)=$	0.00056800

Table 15.:

x
0.00155
0.00155
-0.0014
0.000568

CONCLUSION

Calculations show that a non-nodal point **12** does not influence the adjusted results of a local levelling network, so introducing a non-nodal point into the adjustment of a local levelling network is unnecessary.

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